A Problem in the Free Vibration of Stiffened Cylindrical Shells

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The paper presents an exact solution of the free vibration of a rib stiffened, freely supported, circularly cylindrical shell. The edge forces of the curved panels between stiffeners are written in terms of the edge deformations, which in turn are related geometrically to the centroidal deformations of the ribs. Effects of eccentricity and finite width of the ribs are included. It is shown that the motion of the structure is governed by just four second-order difference equations in terms of the rib deformations as functions of the discrete variable r labelling the ribs. The solution shows that for all modes the deformations are described by trigonometric functions in the axial coordinate. Circumferentially, the rib deformations are governed by elementary discrete trigonometric functions with argument $m\pi r/n$, in which m is an integer. If there are n ribs, there are just $n/2$ unique values of m giving solutions to the difference equations, but for each m there is an infinite number of natural frequencies with different modal functions. The numerical solutions verify the possible importance of rib eccentricity on the natural frequencies. It is further demonstrated that effects of ribs cannot be generalized since they are heavily dependent on number, size, and eccentricity.

Nomenclature

a	= radius of the middle surface of a panel
$\bar{A}$	= cross-sectional area of a stiffener
$\mathbf{A}$	= coefficient matrix of the solutions to the panel equations
b	= stiffener width
B_1^k,B_2^k,C^k	= integration constants
c	= $h/a[1/12(1-\nu^2)]^{1/2}$
$\mathbf{C}$	= vector of constants, C^k
e	= stiffener eccentricity
E,G	= elastic and shear moduli of the material
$\bar{E}$	= Boolean shift operator
$\mathbf{E}$	= coefficient matrix in the force-deformation equations
g_1,g_2,g_3	= parameters in the $\mathbf{G}$ matrix
$\mathbf{G}$	= transformation matrix between the panel edge and the stiffener centroid
h	= panel thickness
$\mathbf{H}$	= $\mathbf{EA}(\varphi)\|_{\varphi=\varphi_1}$
I_x,I_y,I_z	= moments of inertia of a stiffener cross section
$\mathbf{I}$	= matrix with elements $\delta_{ij}(-1)^{i+j}$
j	= axial harmonic index number
k	= elements of $\mathbf{K}$
$\mathbf{K},\mathbf{K}_1,\mathbf{K}^b$	= stiffness matrices of the complete shell, panel, and stiffener
L	= shell length
m	= circumferential index
M,N,P,Q,Q^*,R,S	= panel stress resultants
$\bar{M},\bar{N},\bar{R},\bar{S},\bar{T}$	= stiffener stress resultants
n	= number of stiffeners
r	= stiffener number
s	= root of the panel equations
$\mathbf{S}$	= vector of panel edge forces
t	= time
u,v,w	= panel middle surface deformations
$\mathbf{u}$	= vector of panel middle surface deformations and rotation
U,V,W	= deformations of a stiffener centroid
$\mathbf{U}$	= vector of stiffener centroid deformations and rotation
x,φ,z	= panel coordinates
X,Y,Z	= stiffener coordinates
α_j	= $j\pi a/L$
β,γ	= parameters in the solution to the panel equations
Θ	= axial rotation of a stiffener
κ_Y,κ_Z	= shear constants
λ^2	= $\rho\omega^2a^2(1-\nu^2)/E$
ν	= Poisson's ratio of the material
ρ	= mass density of the material
φ_1	= half panel angular width
Φ	= rotation of a normal to a panel middle surface about the x axis = $(\partial w/\partial\varphi - v)/a$
ω	= circular natural frequency
$\Delta\nabla,\Delta_M$	= difference operators

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Introduction

THE purpose of this paper is to present an exact solution, to the point that beam and shell theory are exact, for the free vibration of a circularly cylindrical shell which is freely supported at its ends and reinforced by ribs of symmetrical cross section. As will be shown, the natural frequencies are determined by the vanishing of a 4×4 determinant the form of which will be presented, and the normal modes are described by the products of discrete and continuous functions. The results are independent of the number and spacing of the ribs except that they must be spaced equally, and therefore the analysis can be useful in evaluating the accuracy and range of validity of other analyses which are based on simplifying assumptions predicted on closely spaced ribs. Furthermore, the results will aid in interpreting the complex spectrum obtained in vibration tests of stiffened shells.

The basis of most analyses has been replacement of the actual structure by an equivalent continuum, that is to say a continuous shell whose properties are chosen so as to represent, at least in a gross sense, those of the original structure. Thus the analysis reduces essentially to choosing a method of averaging the effects of the discrete elements with those of the continuous elements, either by "smearing out" the stiffener properties or by replacing the shell with one made of an orthotropic material. Representative of these are Refs. 1, 2, and 3. For shells with closely spaced stiffeners, a frequency analysis based on an equivalent orthotropic shell could be expected to give reasonable results, as is the case. In the case of shells with only a few stiffeners, the equivalent continuum can no longer represent the true motion of the structure, and the calculated natural frequencies in general are inaccurate. In either case, however, the mode shapes determined in this type of analysis represent the true motion only in a gross sense, and it is not possible to calculate the true state of stress in the vibrating structure, as might be necessary in dynamic response studies, for example.

Some authors have accounted for the discreteness of the stiffeners in an approximate way. Taking a viewpoint opposite to the equivalent continuum, Wah[4] lumped the stiffness effects of the curved panels between stiffeners with those of the stiffeners and obtained a difference equation which he solved for a freely supported boundary. The criticism of the method of the equivalent continuum applies also to that of Wah, in that though the frequencies and nodal displacements are probably accurate, when the results of the theory are applied within its range of validity (according to Wah this is when the motions are primarily radial), they do not permit detailed calculation of the structural behavior. A more recent paper[5] reported an analysis using the Ritz technique that accounted for the discreteness of the stiffeners in an approximate way by calculating the potential energy of the stiffeners using the same functions as for the shell elements, thus forcing compatibility. This analysis is also approximate in that it represents the motion of the shell in the circumferential direction by trigonometric functions, which is not the case as will be shown later. This paper gives an excellent bibliography of the work done on the vibration of shells stiffened by both rings and stringers.

Schnell and Heinrichsbauer[6] presented a numerical procedure based on a transfer matrix approach, the derivation of which was based on Donnell's shell equations. They accounted for the discreteness of the structure correctly by representing the panel motion by the dynamic stiffness matrix. Their method is useful for calculating the vibrational characteristics of nonregular shells, those with nonsimilar beams and panels. The paper by Bartolozzi[7] also analyzes this problem, accounting for the discreteness of the structure, but some highly restrictive assumptions as to the nature of the vibration limit the usefulness of the results. In essence he obtained the frequency equations for two particular types of motion which result from the more general analysis to follow.

A stringer stiffened shell is in reality a composite structure composed of segments of curved panels connected along their straight edges to flexible beams. It has the characteristic that it is continuous in one direction, that parallel to the axis of revolution, with its behavior governed by the differential equations of beam and shell theory, and discrete-continuous in the circumferential direction. The behavior circumferentially is therefore governed by difference equations expressing the interrelation of the discrete beams as well as the differential equations of the curved elements. (More correctly, the dynamics problem is governed by partial difference-differential equations, since the time is also involved, but because the problem under consideration in this paper is one of free vibrations and the time variable is easily separated, this distinction is not important.)

This paper presents a derivation of the equations of the composite structure. Only approximations which are in the theory of beams and shells or approximations the effects of which are known are used. The equations are for the particular case of a freely supported shell, and the coefficients of the frequency determinant, which as will be seen is just of order four, is derived, as is the functional form of the modes. The results of the solution of the vibration of several shells are presented.

Analysis

The derivation of the frequency equation for the total structure proceeds as follows. First the equilibrium equations of a typical stiffening beam are obtained. Included in the equations are the beam elastic and inertia forces, and the forces acting on the beam from the adjoining curved panels. Next, all of the variables in the beam and panel equations will be expressed as trigonometric functions which satisfy the freely supported conditions at each end of the shell. This solution reduces the panel equations to a set of linear, ordinary differential equations with constant coefficients with the angle φ as the independent variable. The solution of these equations is straightforward, which makes it possible to express the forces along an edge φ = constant in terms of the deformations along those edges, which are the intersections of the panel with the adjacent stiffening beams. In turn, the edge deformations are related geometrically to the beam centroidal deformations. Because the edge forces of a particular panel are functions of the deformations of both of its edges, the result of substituting the expressions for the edge forces into the beam equations is a set of difference equations in terms of the beam deformations. Finally, these difference equations are solved for the complete cylinder and the frequency equation is determined.

Rib Equations

The six equilibrium equations of a rib are obtained by balancing the beam stress resultants, panel edge forces, and the inertia forces. For the problem of free vibration, the time derivatives are replaced by $-\omega^2$, the square of the circular natural frequency.

The derivatives of the beam forces are related to deformations by the theories of stretching, twisting, and bending of bars. Substitution of the formulas for a uniform Timoshenko beam[12] including rotatory inertia gives after some rearranging as the beam equations

$$E\bar{A}(\partial^2/\partial X^2 + \rho\omega^2/E)U + S - S' = 0$$

$$-EI_Z\left[\frac{\partial^4}{\partial X^4} + \frac{\rho\omega^2}{E}\left(1 + \frac{E\kappa_Z}{G}\right)\frac{\partial^2}{\partial X^2} - \frac{\rho\omega^2\bar{A}}{EI_Z}\right]V + g_2(P - P') + g_1(Q + Q') + ag_1\frac{\partial(S + S')}{\partial X} = 0$$

$$-EI_Y\left[\frac{\partial^4}{\partial X^4} + \frac{\rho\omega^2}{E}\left(1 + \frac{E\kappa_Y}{G}\right)\frac{\partial^2}{\partial X^2} - \frac{\rho\omega^2\bar{A}}{EI_Y}\right]W + g_2(Q - Q') - g_1(P + P') - e\frac{\partial(S - S')}{\partial X} = 0 \qquad (1)$$

$$GI_X\left(\frac{\partial^2}{\partial X^2} + \frac{\rho\omega^2}{G}\right)\Theta + (eg_2 - g_1^2)(P - P') + \frac{b}{2}(g_2 + g_3)(Q + Q') + R - R' = 0$$

Here, E, G, and ν are the usual elastic constants and κ_Z, κ_Y are the shear constants. The stress resultants and the coordinate systems are defined in Figs. 1 and 2. The primes denote the edge forces acting on the negative Y face of the rib. The mass density of the material is ρ, the radius of the middle surface of the panels is a, the cross-sectional area of a stiffener is $\bar{A}$, t is the time, $g_1 = b/2a$, $g_2 = (1 - g_1^2)^{1/2}$, $g_3 = e/a$, and the moments of inertia of a stiffener about the indicated axes are I_X, I_Y, I_Z. The width, b, and eccentricity, e, of a stiffener are shown in Fig. 3.

Solutions to the beam equations of motion which satisfy a freely supported boundary, defined by $\bar{N} = V = W = \Theta =$

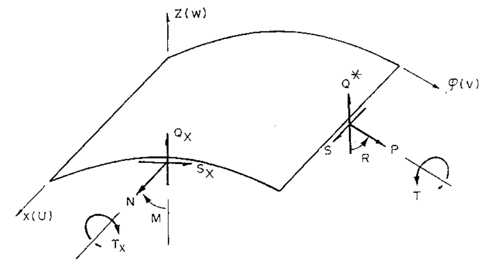

Fig. 1 Panel stress resultants.

$\bar{M} = \bar{R} = 0$ at the ends $X = 0, L$ are

$$\begin{bmatrix} V \\ W \\ \Theta \end{bmatrix} = \begin{bmatrix} V_j \\ W_j \\ \Theta_j \end{bmatrix} \sin\alpha_j \frac{X}{a}, \quad U = U_j \cos\alpha_j \frac{X}{a} \qquad (2)$$

in which $\alpha_j = j\pi a/L$, $j = 1, 2, \ldots$, and L = length. It is anticipated that the shell edge forces are given by

$$\begin{bmatrix} P(x,\varphi) \\ Q(x,\varphi) \\ R(x,\varphi) \end{bmatrix} = \begin{bmatrix} P_j(\varphi) \\ Q_j(\varphi) \\ R_j(\varphi) \end{bmatrix} \sin\alpha_j \frac{x}{a}, \; S(x,\varphi) = S_j(\varphi) \cos\alpha_j \frac{x}{a}$$

With these solutions, Eqs. (1) become

$$EA[-(\alpha_j/a)^2 + \rho\omega^2/E]U_j + S_j - S'_j = 0$$

$$-EI_Z\left[\left(\frac{\alpha_j}{a}\right)^4 - \frac{\rho}{E}\left(\frac{\alpha_j}{a}\right)^2\left(1 + \frac{E\kappa_Z}{G}\right) - \frac{\rho\bar{A}\omega^2}{EI_Z}\right]V_j +$$
$$g_2(P_j - P'_j) + g_1(Q_j + Q'_j) - \alpha_j g_1(S_j + S'_j) = 0$$

$$-EI_Y\left[\left(\frac{\alpha_j}{a}\right)^4 - \frac{\rho}{E}\left(\frac{\alpha_j\omega}{a}\right)^2\left(1 + \frac{E\kappa_Y}{G}\right) - \frac{\rho\bar{A}\omega^2}{EI_Y}\right]W_j -$$
$$g_1(P_j + P'_j) + g_2(Q_j - Q'_j) + g_3\alpha_j(S_j - S'_j) = 0$$

$$GI_X\left[-\left(\frac{\alpha_j}{a}\right)^2 + \frac{\rho\omega^2}{G}\right]\Theta_j + (eg_2 - g_1^2)(P_j - P'_j) +$$
$$\frac{b}{2}(g_2 + g_3)(Q_j + Q'_j) + R_j - R'_j = 0$$

which in matrix notation is

$$\mathbf{K}_j{}^b\mathbf{U}_j + \mathbf{G}_j{}^T\mathbf{S}_j - \mathbf{G'}^T{}_j\mathbf{S}'_j = 0 \qquad (3)$$

in which $\mathbf{U}_j{}^T = [U_jV_jW_j\Theta_j]$, $\mathbf{S}_j{}^T = [S_jP_jQ_jR_j]$, and

$$\mathbf{G}_j = \begin{bmatrix} 1 & \mp\alpha_j g_1 & \alpha_j g_3 & 0 \\ 0 & g_2 & \mp g_1 & eg_2 - g_1^2 \\ 0 & \pm g_1 & g_2 & \pm b(g_2 + g_3)/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When a double sign is written in the previous equation, the lower sign refers to the $\mathbf{G}'$ matrix. Otherwise the signs are the same for both $\mathbf{G}_j$ and $\mathbf{G}'_j$. The three components of deformation and the rotation of the middle surface of a panel at its edge in terms of the deformations of the centroid of the adjoining rib are $\mathbf{G}_j\mathbf{U}_j$ and $\mathbf{G}'_j\mathbf{U}_j$ for the positive and negative Y sides, respectively.

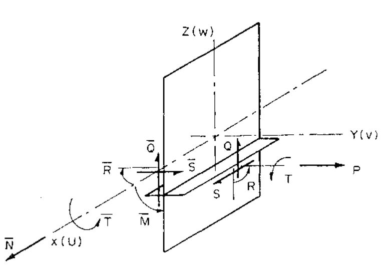

Fig. 2 Rib stress resultants.

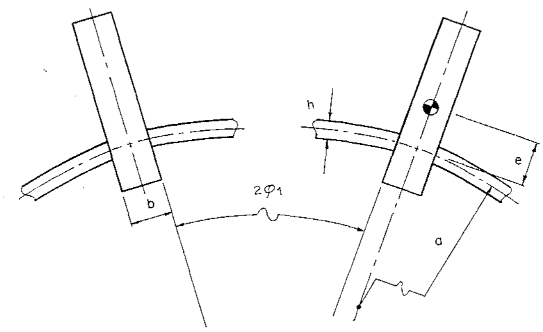

Fig. 3 Stiffener detail.

Panel Equations

The equations of motion of a thin shell in circularly cylindrical coordinates x,φ can be written in terms of the middle surface displacements u, v, w, as a set of linear partial differential equations. It is assumed that the panels are thin so that transverse shear deformations and rotatory inertia effects are negligible, and therefore the set of equations is of eighth order. Donnell's equations[10] were used for the numerical examples presented later in the paper. For completeness these equations are given in the Appendix. What follows here, however, is valid for any thin shell theory based on these assumptions. Particular solutions that satisfy the freely supported boundary conditions $N = v = w = M = 0$ at the ends $x = 0, L$ are

$$u(x,\varphi) = u_j(\varphi)\cos\alpha_j(x/a)$$
$$v(x,\varphi) = v_j(\varphi)\sin\alpha_j(x/a)$$
$$w(x,\varphi) = w_j(\varphi)\sin\alpha_j(x/a)$$

Substitution of these solutions into the panel equilibrium equations shows that the motion uncouples into harmonics in the x coordinate direction. The modal solutions can therefore be obtained for each harmonic individually. The result is a set of linear, homogeneous, ordinary differential equations for the jth harmonic in terms of the displacements $u_j(\varphi)$, $v_j(\varphi)$, $w_j(\varphi)$. These equations contain explicitly the frequency ω as a parameter. The solution is straightforward, of the form

$$[u_j(\varphi)v_j(\varphi)w_j(\varphi)] = [B_{1j}B_{2j}C_j]\exp(s\varphi) \qquad (4)$$

in which B_{1j}, B_{2j}, C_j are constants. This substitution gives a homogeneous set of algebraic equations whose determinant must vanish. The result is an eighth-order polynomial in the complex roots, s, containing as parameters the frequency, ω, the term α_j, and the properties of the shell. The eight roots, s, give eight solutions for the displacements, which can be written as

$$[u_j(\varphi)v_j(\varphi)w_j(\varphi)] = \sum_{k=1}^{8}[B_{1j}{}^kB_{2j}{}^kC_j{}^k]\exp(s_k\varphi)$$

There are just eight independent constants, and it is convenient to express the solution in terms of one set of the constants, say the $C_j{}^k$, by substituting Eq. (4) into the panel equations and solving for $B_{1j}{}^k$ and $B_{2j}{}^k$ in terms of $C_j{}^k$. The result for the displacements is

$$\mathbf{u}(\varphi) = \mathbf{A}(\varphi)\mathbf{C} \qquad (5)$$

in which $\mathbf{C}$ is the 8×1 vector of independent integration constants and

$$\mathbf{u}^T(\varphi) = [u(\varphi)v(\varphi)w(\varphi)\Phi(\varphi)]$$

where Φ is the rotation of an element normal to the middle surface of the panel. In these equations and hereafter the harmonic index j is omitted for simplicity. The elements of the **A** matrix are trigonometric and hyperbolic, obtained directly from the exponential form.

Equation (5) gives the variation in φ of the displacements for the jth harmonic in terms of the integration constants C_j^k for a typical panel between stiffeners. Because of the symmetry of a panel, the solution is composed of a symmetric part and an antisymmetric part. In evaluating the forces induced by the shell on the stiffening ribs in terms of the rib deformations, an almost impossible algebraic problem is made tractable by considering symmetric and antisymmetric motions separately. The total forces and motion are the superposition of these two effects.

Symmetrical deformations are defined as those which are physically symmetrical about the x axis of the panel. For symmetrical motion, the formulas for u and w are symmetric functions, those for v and φ are antisymmetric. Formulas for the deformation components for symmetric motion are obtained from Eq. (5) by setting equal to zero the coefficients of antisymmetrical components of w. The four remaining constants can be expressed in terms of the deformations at the edges of the panels from $\mathbf{u}_s(\varphi) = \mathbf{A}_s(\varphi)\mathbf{C}_s$ by $\mathbf{C}_s = \mathbf{A}'^{-1}{}_s\mathbf{u}'_s$. The convention introduced here is that for a variable $f(\varphi)$, the functional notation refers to the variable at an arbitrary angle φ, while the notation f, f' refers to the value of the function at $\varphi = -\varphi_1$ and $\varphi = \varphi_1$, respectively. The panel displacements in terms of the deformations at the edge $\varphi = \varphi_1$ are $\mathbf{u}_s(\varphi) = \mathbf{A}_s(\varphi)\mathbf{A}'^{-1}{}_s\mathbf{u}'_s$. Because of symmetry of the motion, the deformations at the edges can be written as $\mathbf{u}'_s = (\mathbf{u}' + \mathbf{I}\mathbf{u})/2$. The matrix **I** is a 4×4 diagonal with elements $\delta_{ij}(-1)^{i+j}$, where δ_{ij} is the Kronecker delta. In the interior of the panel the deformations are

$$\mathbf{u}_s(\varphi) = \mathbf{A}_s(\varphi)\mathbf{A}'^{-1}{}_s(\mathbf{u}' + \mathbf{I}\mathbf{u})/2 \tag{6}$$

Equations of the Stiffened Shell

Now number the n stiffeners $r = 0,1,2, \ldots, n-1$ and designate as the rth panel that joining ribs r and $r+1$. The equations developed to this point apply to a generic stiffener and panel, but in reality the deformations and forces of the stiffener are functions of the discrete variable r. For clarity the functional notation $\mathbf{U}(r)$ will be introduced, with similar notation for the forces.

The **G** matrices relate the panel edge deformations to the centroidal deformations of the adjoining stiffener as $\mathbf{u}'(r) = \mathbf{G}'\mathbf{U}(r+1)$ and $\mathbf{u}(r) = \mathbf{G}\mathbf{U}(r)$, from which and Eq. (6)

$$\mathbf{u}_s(\varphi;r) = \mathbf{A}_s(\varphi)\mathbf{A}'^{-1}{}_s[\mathbf{G}'\mathbf{U}(r+1) + \mathbf{I}\mathbf{G}\mathbf{U}(r)]/2 \tag{7}$$

In summary, Eq. (7) expresses the symmetric part of the free vibration panel deformations, for the jth x coordinate harmonic, in terms of the centroidal deformations of the stiffening beams lying along the panel boundaries.

Finally required for substitution into the beam equilibrium equations are the panel stress resultants expressed as functions of $\mathbf{U}(r)$. Write the force-deformation equations as

$$\mathbf{S}(\varphi;r) = \mathbf{E}\mathbf{u}(\varphi;r) \tag{8}$$

The forces in symmetric motion follow from Eqs. (7) and (8). With $\mathbf{H}_s \equiv \mathbf{E}\mathbf{A}_s(\varphi)|_{\varphi=\varphi_1}$, the edge forces on the negative face of the rth stiffener are $\mathbf{S}'_s(r-1) = \mathbf{H}_s\mathbf{A}'^{-1}{}_s[\mathbf{G}'\mathbf{U}(r) + \mathbf{I}\mathbf{G}\mathbf{U}(r-1)]/2$. The positive face edge forces on the rth stiffener are, because of the symmetry of the motion, $\mathbf{S}_s(r) = -\mathbf{I}\mathbf{S}'_s(r)$.

Through similar calculations, one computes the deformation and forces for the antisymmetric component of the motion as

$$\mathbf{u}_a(\varphi;r) = \mathbf{A}_a(\varphi)\mathbf{A}'^{-1}{}_a[\mathbf{G}'\mathbf{U}(r+1) - \mathbf{I}\mathbf{G}\mathbf{U}(r)]/2 \tag{9}$$

$$\mathbf{S}'_a(r-1) = \mathbf{H}_a\mathbf{A}'^{-1}{}_a[\mathbf{G}'\mathbf{U}(r) - \mathbf{I}\mathbf{G}\mathbf{U}(r-1)]/2$$

$$\mathbf{S}_a(r) = \mathbf{I}\mathbf{S}'_a(r)$$

The total forces are the sum of the symmetric and antisymmetric components. Substituting the force expressions into the beam equilibrium equations, Eqs. (3), gives the equation of motion in terms of the vector of centroidal deformations of the rth stiffener and the two adjacent stiffeners. With the definitions $\mathbf{K}_s = \mathbf{G}'^T\mathbf{H}_s\mathbf{A}'^{-1}{}_s\mathbf{G}'$ and $\mathbf{K}_a = \mathbf{G}'^T\mathbf{H}_a\mathbf{A}'^{-1}{}_a\mathbf{G}'$ and the identity $\mathbf{G}' = \mathbf{I}\mathbf{G}\mathbf{I}$, the equilibrium equation can be put in the following form:

$$\mathbf{K}^b\mathbf{U}(r) + \tfrac{1}{2}\{(\mathbf{K}_s - \mathbf{K}_a)\mathbf{I}\mathbf{U}(r-1) + [(\mathbf{K}_s + \mathbf{K}_a) + \mathbf{I}(\mathbf{K}_s + \mathbf{K}_a)\mathbf{I}]\mathbf{U}(r) + \mathbf{I}(\mathbf{K}_s - \mathbf{K}_a)\,\mathbf{U}(r+1)\} = 0$$

The products $(\mathbf{H}_s\mathbf{A}'^{-1}{}_s)$ and $(\mathbf{H}_a\mathbf{A}'^{-1}{}_a)$ are symmetric, since they are the stiffness matrices for the panel for symmetric and antisymmetric motion. The $\mathbf{K}_s$ and $\mathbf{K}_a$ matrices are therefore symmetric and $k_{ij}{}^s = k_{ji}{}^s$, $k_{ij}{}^a = k_{ji}{}^a$, where $k_{ij}{}^s$, $k_{ij}{}^a$ are the elements of $\mathbf{K}_s$ and $\mathbf{K}_a$.

Following Dean,[8] it is convenient to introduce a symmetric and an antisymmetric difference operator, defined in terms of the Boole shift operator[9] $\bar{E}^if(r) = f(r+i)$ by $\Delta\nabla = \bar{E}^{-1} - 2 + \bar{E}$ and $\Delta_M = (-\bar{E}^{-1} + \bar{E})/2$. With these operators the equations of motion become

$$(\mathbf{K}_1 + \mathbf{K}^b)\mathbf{U}(r) = 0 \tag{10}$$

The operator matrix, $\mathbf{K}_1$, is defined in the Appendix.

Equation (10) forms an eighth-order set of ordinary difference equations in terms of the stiffener deformations, $\mathbf{U}(r)$, and the frequency, ω. The difference equations are linear, and because the structure is regular, that is the panels and stiffeners are all similar, they have constant coefficients. The general solution for a segment of a shell with arbitrary boundaries along two generators is straightforward, though algebraically involved, and results in eight independent constants[9] of summation. In this particular case, in which the structure is a complete cylinder, periodicity in the circumferential coordinate, r, makes the solution particularly easy.

Let

$$\begin{aligned} U(r) &= U^* \cos(2\pi m/n)r \\ V(r) &= V^* \sin(2\pi m/n)r \\ W(r) &= W^* \cos(2\pi m/n)r \\ \Theta(r) &= \Theta^* \sin(2\pi m/n)r \end{aligned} \qquad m, r = 0,1,2, \ldots, n-1 \tag{11}$$

Substituting these solutions into Eq. (10) and carrying out the indicated difference operations[9] give four homogeneous algebraic equations for the amplitude of the motion, $\mathbf{U}^*$. Write these equations as $(\mathbf{K}_{1m} + \mathbf{K}^b)\mathbf{U}^* = 0$. Then, setting the determinant of the coefficients of these equations equal to zero,

$$|\mathbf{K}_m| = |\mathbf{K}_{1m} + \mathbf{K}^b| = 0 \tag{12}$$

determines the frequency of vibration. The matrix $\mathbf{K}_{1m}$ is given in the Appendix.

For a particular frequency, Eqs. (10) and (11) determine the modal functions of the ribs in terms of one deformation chosen arbitrarily. The modal deformations of a panel, say the rth lying between ribs r and $r+1$, follow from summing Eqs. (7) and (9). Carrying out this operation and using the solutions of Eq. (11) give for the modal functions of the rth panel

$$\mathbf{u}(\varphi;r) = \left[\mathbf{A}_s(\varphi)\mathbf{A}_s{}^{-1}\mathbf{G}'\mathbf{T}_1 \cos\frac{2\pi m}{n}\left(r + \tfrac{1}{2}\right) + \mathbf{A}_a(\varphi)\mathbf{A}_a{}^{-1}\mathbf{G}'\mathbf{T}_2 \sin\frac{2\pi m}{n}\left(r + \tfrac{1}{2}\right)\right]\mathbf{U}^* \tag{13}$$

in which $\mathbf{T}_1$ and $\mathbf{T}_2$ are 4×4 diagonal matrices, the first with terms cos, sin, cos, sin, and the second with terms $-$sin, cos, $-$sin, cos, all functions having the argument $\pi m/n$.

The geometry of the panels and stiffening beams and the material properties determine the vibration characteristics

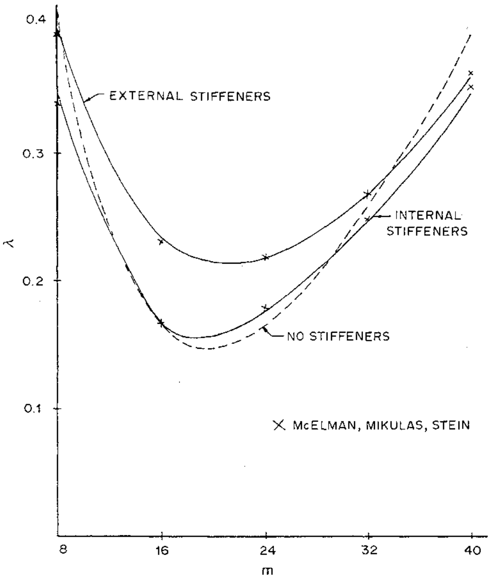

Fig. 4 Frequency vs *m* for a shell with many stiffeners.

of the rib stiffened shell analyzed, the natural frequencies of which are the roots of Eq. (12). Two integer parameters appear in this equation: the integer j, defined by Eq. (2), which prescribes the continuous x coordinate variation of the deformations, and the integer m which prescribes the circumferential variation of the deformation of the centroids of the stiffening beams. For a pair of integers, j,m there is an infinite number of vibration frequencies, the modes of which vary both in the modal ratios determined by $\mathbf{K}_m\mathbf{U}^* = 0$ and in the number of waves in the panel deformations between adjacent stiffeners. In determining the frequency spectrum for a cylinder, one choses the integers j,m and solves for the frequencies required. By systematically varying j,m one arrives at the complete frequency spectrum in the range of interest.

The procedure for finding a particular frequency is as follows. Assume a value for ω and solve for the roots s of the frequency polynomial of the panel. Next, calculate the elements of the frequency equation, Eq. (12). Because of the symmetry of $\mathbf{K}_m$ there are just ten different terms. Now compute $|\mathbf{K}_m|$. If $|\mathbf{K}_m| = 0$ the assumed value of ω is a natural frequency; if not make another choice and repeat the process.

Discussion and Conclusions

The form of Eq. (12) shows that except for the eigenvalue, ω, the geometric and material properties of the shell elements and the integers j and m specify the coefficients of the matrix. The integer j is the number of half waves in the modal deformation in the axial direction (equivalently, the number of antinodes); m is the number of complete circumferential waves on a plot of the rib deformation. The modal vector for each eigenvalue in general is unique, so that though for a given integer pair the variation of the components of stiffener displacement is the same, expressed by Eqs. (11), the relative amplitudes are different. In the usual case, the lowest frequency is associated with relatively large radial displacement compared to the tangential components. Also, for each of these eigenvalues the motion of the panels between stiffeners differs. For example, for the case in which $m = 0$, for which all stiffeners deform identically, the panels move with a different number of nodes between adjoining ribs for each of the different frequencies in the spectrum.

In comparing the present analysis with those using an equivalent continuum or "smeared out" stiffener properties, it is important to keep in mind that the circumferential mode number as used in this analysis is not the number of waves in the panels, such as might be observed in a vibration test, but is the number of waves in the deformation pattern of the ribs. For example, in Ref. 1 the radial deformation, in the notation of the present paper, is taken as

$$w(x,\varphi) = C \sin(j\pi x/L) \cos m\varphi$$

in which case m is the number of waves in the deformation pattern of the skin. The true motion of the panels is not just trigonometric but is trigonometric and hyperbolic, so that the assumption of trigonometric waves in the panel deformation pattern is not correct. When there are many closely spaced ribs, the gross behavior is similar to that of a continuous shell, and the integer m is then approximately equal to the number of circumferential waves.

Usually, the important frequencies are in the low range, and therefore determining the minimum frequency assumes importance. The lower frequencies always occur for a single half wave modal function in the axial direction ($j = 1$). The value of m which gives the lowest frequency depends on the shell geometry, and it is not possible to determine from the characteristic equation just which value of m gives a minimum frequency, other than by calculating the lowest frequency for a range of m values and then selecting the absolute minimum of these. Again, if there are many stiffeners the problem is simplified because of the similarity between the reinforced shell behavior and that of an unstiffened shell, since the value of m which gives the minimum frequency is very close to or equal to the value giving the minimum frequency of the unstiffened shell, and therefore results of an analysis of an unreinforced shell can be used to obtain a first estimate of the appropriate value of m.

The solution of the difference equations is valid for the range $m = 0,1,\ldots, n-1$, as indicated in Eq. (11). Inspection of Eq. (12) shows, however, that an eigenvalue calculated for a value of $m' = n - m$ is the same as that calculated for m. Therefore, if the k_{ij} are the elements of the $\mathbf{K}_m$ matrix for an assumed eigenvalue (not necessarily correct) calculated for m, and k'_{ij} are the elements for the same assumed eigenvalue calculated for m', then $k_{ij} = k'_{ij}(-1)^{i+j}$. Thus, the $\mathbf{K}'_m$ matrix can be obtained from the $\mathbf{K}_m$ matrix by an even number of successive sign changes of rows and columns, which means that $|\mathbf{K}_m| = |\mathbf{K}'_m|$. Therefore, the same value of ω causes each to vanish and they have the same eigenvalues. Further, components of the modal vectors are related by $U^{*\prime} = U^*$, $V^{*\prime} = -V^*$, $W^{*\prime} = W^*$, $\Theta^{*\prime} = -\Theta^*$. In words, the eigenvalues calculated for m and m' are equal, and the eigenvector for m' is the same as that for m with the shell turned end to end. This is a result of the rotational symmetry of the shell and indicates that there is no preferred direction for the motion. This is not the case in general, for example for a stiffened sheet which does not form a closed shell. Equation (10) is an eighth-order set of difference equations, and therefore there are eight independent summation constants in the solution whose values depend on the boundaries along the edges parallel to the ribs. The solutions are discrete trigonometric and hyperbolic functions without the property of repeating for $m' = n - m$. A practical consequence is that in defining the frequency spectrum for a complete shell, the eigenvalues must be calculated only for the range $m = 0,1,2,\ldots, n/2$.

Two special cases of interest are the rotationally symmetric modes, $m = 0$, and the antisymmetric modes, $m = n/2$. Setting $m = 0$ in Eq. (11) and expanding the determinant give $[k_{11}{}^s k_{33}{}^s - (k_{13}{}^s)^2][k_{22}{}^a k_{44}{}^a - (k_{24}{}^a)^2] = 0$. Vanishing of the first factor gives the frequency corresponding to radial and axial motion only with all ribs moving in phase, with no tangential deformation or rotation of the ribs. The second factor vanishes for an eigenvalue corresponding to tangential deformation and rotation only. For each of these types of motion, however, the panels undergo motion in every coordinate.

The frequency equation for the second case, $m = n/2$, is $[k_{11}{}^a k_{33}{}^a - (k_{13}{}^a)^2][k_{22}{}^s k_{44}{}^s - (k_{24}{}^s)^2] = 0$. The motion here is uncoupled as in the first case except that adjacent ribs are out of phase by 180°. These are the particular types of motion studied in Ref. 7.

It should be noted in passing that Eq. (10) is not limited to cylindrical panels between ribs, but is valid for any panels which are surfaces of translation, including as special cases plates and membranes, and which are freely supported at the ends. For all but the simplest forms, the stiffness matrices $\mathbf{K}_s$ and $\mathbf{K}_a$ would have to be generated numerically.

In implementing Eq. (12) for solution, Donnell's equations[10] were used in deriving the terms $k_{ij}{}^s$ and $k_{ij}{}^a$. These equations are displayed in the Appendix. The further approximation of eliminating the in-plane inertia terms from the panel equations reduced the eighth-order polynomial for the roots, s, of the panel equations to a quartic in s^2 which could be solved explicitly. The in-plane inertia of the panels was accounted for approximately by replacing the beam mass in the U, V equations by the term $\rho(\bar{A} + 2ah\varphi_1)$. In other words, in-plane inertia effects were lumped, while radial inertia was considered distributed. Finally, the shear deformation and rotatory inertia terms were eliminated from the beam equations. These approximations limited the range of validity to thin shells, slender beams, and modes with many circumferential waves. This includes an important class of problems, but extension to other problems by use of a more accurate shell theory would be relatively simple, the most difficult obstacle being determining the roots s for each assumed value of the frequency if in-plane inertia effects were retained.

Two examples, one with closely spaced stiffeners and the other with only a few stiffeners, are given to illustrate the results of the study. These were calculated on the IBM operating system 360, at the Triangle Universities Computation Center, Research Triangle Park, N. C. Each solution required less than five seconds. The first, which has 200 ribs, is the configuration studied in Ref. 1. Figure 4 shows the frequency of vibration as a function of the circumferential mode number, m, which has approximately the same meaning as in the reference since the ribs are closely spaced. The curves are obviously valid only for integer values of m. The data given in the reference were used except for the value of ρ which was not given. A value of $\rho = 0.259 \times 10^{-3}$ lb. sec^2/in.^{4}, typical for aluminum, was assumed. The higher curve in Fig. 4 gives the frequencies for externally mounted stiffeners. The considerable lowering of the frequency resulting from internal mounting of the ribs is shown by the lower curve, particularly in the region of the minimum frequency. Some frequencies from the reference are shown as points in the figure. These were taken from a small drawing and contain some error. The dashed curve gives the frequencies of the same shell without

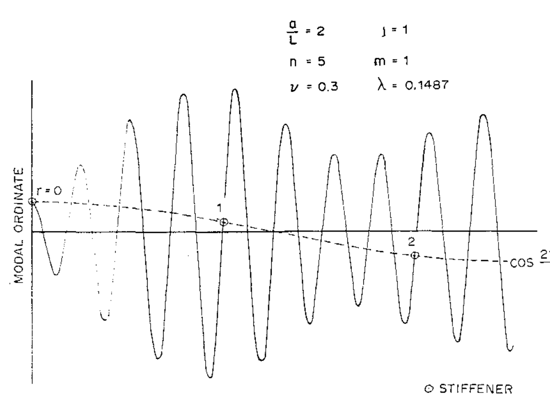

Fig. 5 Radial mode shape.

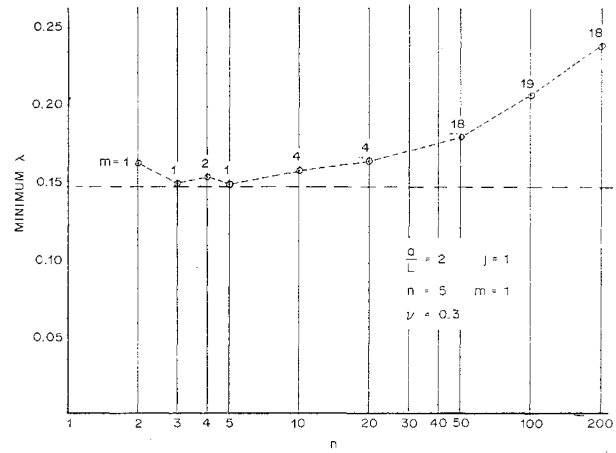

Fig. 6 Frequency spectrum for a shell with few stiffeners.

stiffeners. For the higher frequencies, the modes of which have many circumferential waves, the addition of stiffeners lowers the frequencies, apparently because the added mass is more dominant than the additional stiffness. It is possible to lower the entire spectrum by adding ribs, the extreme case being concentric ribs with essentially no bending stiffness.

The second example is a shell with just five stiffeners. The radial mode shape corresponding to the lowest natural frequency ($\lambda = 0.1487$, $m = 1$, $j = 1$) is shown in Fig. 5. Only half the mode is shown since the shell and mode are symmetric about the diameter passing through the zeroth rib. In the figure, the continuous curve is the radial deformation of the middle surface and the dashed curve is the solution to the rib deformation, $\cos(2\pi mr/n)$. Note that there are 19 waves in the modal pattern in the panels, whereas for the same shell with no ribs, the frequency is only slightly less ($\lambda = 0.1468$) and there are 18 circumferential waves in the deformation pattern. Thus, the ribs, for this particular case, serve mainly to modify the amplitude of the deformation, resulting in the curve shown in Fig. 5 with the beat type pattern. Figure 6 shows the effect of the change in the minimum frequency caused by a variation in the number of ribs on the shell of the second example. The erratic nature of the shell behavior due to stiffening is indicated by the lower end of the spectrum as well as by the fact that there is no pattern as to at just what value of m the minimum frequency occurs.

In summary, this paper has presented an analysis of the free vibration of a freely supported circularly cylindrical shell which is reinforced by equally spaced ribs. The analysis is exact to the point that beam and thin shell theories are exact, which is to say there are no approximations such as lumping of mass, smearing of rib effects, or replacing the actual shell with an approximating continuum such as an orthotropic material. In the examples the in-plane inertia effects were lumped, but though this greatly simplifies the numerical work it is not imperative in the analysis. It is possible to reduce the order of the panel equations by using membrane shell theory.[11] This is of very little help, however, except in the case $m = 0$, since the resulting equations still require digital computer calculation. It is possible with a membrane theory to obtain a formula in the $m = 0$ case for the frequencies, but this is of little value since it is the bending that establishes the minimum frequency.

Appendix

In terms of displacements, Donnell's equations[10] describing the free vibration of a cylindrical panel are

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1+\nu}{2a}\frac{\partial^2 v}{\partial x \partial \varphi} + \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{1}{a^2}\lambda^2 u = 0$$

$$\frac{1+\nu}{2a}\frac{\partial^2 u}{\partial x \partial \varphi} + \frac{1}{a^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2}\frac{\partial w}{\partial \varphi} + \frac{1}{a^2}\lambda^2 v = 0$$

$$\frac{\nu}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial \varphi} + \frac{1}{a^2} w + \frac{h^2}{12}\left(\frac{\partial^4 w}{\partial x^4} + \frac{2}{a^2}\frac{\partial^4 w}{\partial x^2 \partial \varphi^2} + \frac{1}{a^4}\frac{\partial^4 w}{\partial \varphi^4}\right) - \frac{1}{a^2}\lambda^2 w = 0$$

Where $\lambda^2 = \rho\omega^2 a^2(1-\nu^2)/E$. Ignoring the in-plane inertia components and substituting the solutions of Eq. (4) enable one to write the polynomial for the roots s as $(h/a)^2(s^2-\alpha^2)^4/12 + (1-\nu^2)\alpha^4 - \lambda^2(s^2-\alpha^2)^2 = 0$. The eight roots are, with $c^2 = (h/a)^2/12(1-\nu^2)$,

$$s = \pm\left(\frac{\pm\lambda}{c[2(1-\nu^2)]^{1/2}}\left(1 \pm \left\{1 - \left[\frac{2c\alpha^2(1-\nu^2)}{\lambda^2}\right]^2\right\}^{1/2}\right)^{1/2} + \alpha^2\right)^{1/2}$$

Now number the roots so that s_1, s_2, s_3, s_4, are real, the remainder imaginary. The solution matrices can now be written as

$$\mathbf{A}_s = \begin{bmatrix} \gamma_1 \cosh\bar{s}_1\varphi & \gamma_3 \cosh\bar{s}_3\varphi & \gamma_5 \cos\bar{s}_5\varphi & \gamma_7 \cos\bar{s}_7\varphi \\ \bar{\beta}_1 \sinh\bar{s}_1\varphi & \bar{\beta}_3 \sinh\bar{s}_3\varphi & -\bar{\beta}_5 \sin\bar{s}_5\varphi & -\bar{\beta}_7 \sin\bar{s}_7\varphi \\ \cosh\bar{s}_1\varphi & \cosh\bar{s}_3\varphi & \cos\bar{s}_5\varphi & \cos\bar{s}_7\varphi \\ (1/a)(\bar{s}_1 - \bar{\beta}_1)\sinh\bar{s}_1\varphi & (1/a)(\bar{s}_3 - \bar{\beta}_3)\sinh\bar{s}_3\varphi & -(1/a)(\bar{s}_5 - \bar{\beta}_5)\sin\bar{s}_5\varphi & -(1/a)(\bar{s}_7 - \bar{\beta}_7)\sin\bar{s}_7\varphi \end{bmatrix}$$

$$\mathbf{A}_a = \begin{bmatrix} \gamma_1 \sinh\bar{s}_1\varphi & \gamma_3 \sinh\bar{s}_3\varphi & \gamma_5 \sin\bar{s}_5\varphi & \gamma_7 \sin\bar{s}_7\varphi \\ \bar{\beta}_1 \cosh\bar{s}_1\varphi & \bar{\beta}_3 \cosh\bar{s}_3\varphi & \bar{\beta}_5 \cos\bar{s}_5\varphi & \bar{\beta}_7 \cos\bar{s}_7\varphi \\ \sinh\bar{s}_1\varphi & \sinh\bar{s}_3\varphi & \sin\bar{s}_5\varphi & \sin\bar{s}_7\varphi \\ (1/a)(\bar{s}_1 - \bar{\beta}_1)\cosh\bar{s}_1\varphi & (1/a)(\bar{s}_3 - \bar{\beta}_3)\cosh\bar{s}_3\varphi & (1/a)(\bar{s}_5 - \bar{\beta}_5)\cos\bar{s}_5\varphi & (1/a)(\bar{s}_7 - \bar{\beta}_7)\cos\bar{s}_7\varphi \end{bmatrix}$$

in which $\beta_k = s_k[\alpha^2(2+\nu) - s_k^2]/(s_k^2 - \alpha^2)^2$, $\gamma_k = \alpha(s_k^2 + \nu\alpha^2)/(s_k^2 - \alpha^2)^2$, and $\bar{s}_k = s_k$, $\bar{\beta}_k = \beta_k$ for $k = 1,2,3,4$ and $\bar{s}_k = -(-1)^{1/2}s_k$, $\bar{\beta}_k = -(-1)^{1/2}\beta_k$ for $k = 5,6,7,8$. The **E** matrix for the jth harmonic is

$$\mathbf{E} = \frac{Eh}{1-\nu^2}\begin{bmatrix} \frac{1-\nu}{2a}\frac{d}{d\varphi} & \frac{1-\nu}{2a}\alpha & 0 & 0 \\ -\frac{\nu\alpha}{a} & \frac{1}{a}\frac{d}{d\varphi} & \frac{1}{a} & 0 \\ 0 & 0 & -\left[\frac{h^2}{12a^3}\frac{d^3}{d\varphi^3} - (2-\nu)\alpha^2\frac{d}{d\varphi}\right] & 0 \\ 0 & 0 & \frac{h^2}{12a^2}\left(\frac{d^2}{d\varphi^2} - \nu\alpha^2\right) & 0 \end{bmatrix}$$

The $\mathbf{K}_1$ and $\mathbf{K}_{1m}$ matrices appearing in Eqs. (10) and (12) are

$$\mathbf{K}_{1m} = \begin{bmatrix} \bar{k}_{11}\sin^2\mu - k_{11}^s & \bar{k}_{12}\sin 2\mu & \bar{k}_{13}\sin^2\mu - k_{13}^s & \bar{k}_{14}\sin 2\mu \\ \bar{k}_{12}\sin 2\mu & -\bar{k}_{22}\sin^2\mu - k_{22}^a & \bar{k}_{23}\sin 2\mu & -\bar{k}_{24}\sin^2\mu - k_{24}^a \\ \bar{k}_{13}\sin^2\mu - k_{13}^s & \bar{k}_{23}\sin 2\mu & \bar{k}_{33}\sin^2\mu - k_{33}^s & \bar{k}_{34}\sin 2\mu \\ \bar{k}_{14}\sin 2\mu & -\bar{k}_{24}\sin^2\mu - k_{24}^a & \bar{k}_{34}\sin 2\mu & -\bar{k}_{44}\sin^2\mu - k_{44}^a \end{bmatrix}$$

$$\mathbf{K}_1 = \frac{1}{2}\begin{bmatrix} \bar{k}_{11}\Delta\nabla + 4k_{11}^s & -2\bar{k}_{12}\Delta_M & \bar{k}_{13}\Delta\nabla + 4k_{13}^s & -2\bar{k}_{14}\Delta_M \\ 2\bar{k}_{12}\Delta_M & -\bar{k}_{22}\Delta\nabla + 4k_{22}^a & 2\bar{k}_{23}\Delta_M & -\bar{k}_{24}\Delta\nabla + 4k_{24}^a \\ \bar{k}_{13}\Delta\nabla + 4k_{13}^s & -2\bar{k}_{23}\Delta_M & \bar{k}_{33}\Delta\nabla + 4k_{33}^s & -2\bar{k}_{34}\Delta_M \\ 2\bar{k}_{14}\Delta_M & -\bar{k}_{24}\Delta\nabla + 4k_{24}^a & 2\bar{k}_{34}\Delta_M & -\bar{k}_{44}\Delta\nabla + 4k_{44}^a \end{bmatrix}$$

where $\bar{k}_{ij} = k_{ij}^s - k_{ij}^a$ and $\mu = \pi m/n$.

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